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Path independence and charge quantization

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Abstract. It is shown that if quantum electrodynamics is assumed to be manifestly gauge independent and path independent then electric charge must be quantized.

1. Introduction

Mandelstam (1962) has formulated a manifestly gauge independent quantum electrodynamics in which the electromagnetic vector potential A_μ never need be introduced. The gauge invariant field variables at x for a spinless particle with charge q_1 are

$$\Phi(x, \mathrm{P}) = \phi(x)\exp\left(-\mathrm{i}q_1 \int_{\mathrm{P}\ -\infty}^{x} \mathrm{d}\xi_\mu A_\mu(\xi)\right) \tag{1}$$

and

$$\Phi^*(x, \mathrm{P}) = \phi^*(x)\exp\left(+\mathrm{i}q_1 \int_{\mathrm{P}\ -\infty}^{x} \mathrm{d}\xi_\mu A_\mu(\xi)\right), \tag{2}$$

where the integrals are taken over the space-like path P from $-\infty$ to the field point x. The operators ϕ and ϕ^* satisfy the usual equations

$$\left[\left(\frac{\partial}{\partial x_\mu} - \mathrm{i}q_1 A_\mu\right)^2 - m^2\right]\phi = 0 \tag{3}$$

and

$$\left[\left(\frac{\partial}{\partial x_\mu} + \mathrm{i}q_1 A_\mu\right)^2 - m^2\right]\phi^* = 0. \tag{4}$$

We use natural units with $\hbar = c = 1$. The choice of $-\infty$ as the 'fixed point' from which phases are calculated is arbitrary. Any other point would do as well.

The price that one must pay for manifest gauge independence is the path dependence in (1) and (2). Mandelstam argues that the path dependence is fundamentally connected with the arbitrariness in the choice of phase factors in the operators of charged fields. This path dependence which renders the field variables non-local is a very unpleasant aspect of the theory. A formalism which is both manifestly gauge independent and path independent would be more desirable. We would like to show that requiring the theory to be path independent gives results consistent with experiment and provides a very natural explanation of charge quantization.

2. Path independence

Consider a charge q_1 associated with the field $\Phi(x, P)$. Let P_1 and P_2 be two paths to the field point x from $-\infty$. These paths are the same everywhere except between the points x_1 and x_2 where they are separate (see figure 1). Path C is the closed path $P_2 - P_1$. The

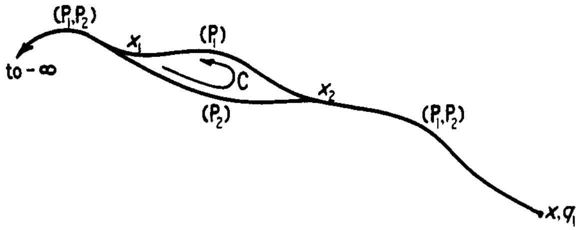

Figure 1. Diagram showing the closed path $C \equiv P_2 - P_1$ and the paths P_1 and P_2. The charge q_1 is associated with the field $\Phi(x, P)$. Various segments of the curve are labelled with symbols in parentheses denoting those paths which follow that segment. x_1, x_2, and x denote points.

theory will be path independent if and only if

$$\Phi(x, P_1) = \Phi(x, P_2) \tag{5}$$

for all space-like P_1 and P_2. We could alternatively require

$$\Phi^*(x, P_1) = \Phi^*(x, P_2) \tag{6}$$

which would lead to the same results below. Now

$$\exp\left(-iq_1 \int_{P_2\, -\infty}^{x} d\xi_\mu A_\mu(\xi)\right) = \exp\left(-iq_1 \int_{P_1\, -\infty}^{x} d\xi_\mu A_\mu(\xi)\right) \exp\left(-iq_1 \oint_C d\xi_\mu A_\mu(\xi)\right), \tag{7}$$

where the exponential can be written in factored form on the right of (6) from the Campbell (1898), Baker (1902, 1903, 1904), Hausdorff (1906) theorem and the fact that

$$[\Phi(x, P_1), \Phi(y, P_2)] = 0 \tag{8}$$

for x–y non-timelike and arbitrary space-like paths P_1 and P_2 from Mandelstam (1962). (8) implies in particular

$$\left[\int_{P_1\, -\infty}^{x} A_\mu(\xi)\, d\xi_\mu, \int_{P_2\, -\infty}^{x} A_\nu(\xi)\, d\xi_\nu\right] = 0 \tag{9}$$

from which (7) follows. From (7) we see that (5) will hold if and only if

$$\exp\left(-iq_1 \oint_C d\xi_\mu A_\mu(\xi)\right) = 1. \tag{10}$$

Alternatively following Cabibbo and Ferrari (1962) we can *define* the path dependence of the field quantities in terms of the electromagnetic field tensor $F_{\mu\nu}$ as

$$\Phi(x, P_2) = \Phi(x, P_1) \exp\left(\frac{-iq_1}{2} \int_S F_{\mu\nu}\, d\sigma^{\mu\nu}\right) \tag{11}$$

where S is a surface delimited by the closed path C. An application of the relativistic generalization of the Stokes theorem then leads to (10) directly if we are to have a path independent formalism.

Now if A_μ is a classical field (10) holds if and only if

$$q_1 \oint_C \mathrm{d}\xi_\mu A_\mu(\xi) = \pm 2\pi n \tag{12}$$

where n is an integer. If A_μ is a quantum operator, (10) is satisfied in operator form if all the eigenvalues of $\oint_C \mathrm{d}\xi_\mu A_\mu(\xi)$ are integer multiples of $2\pi/q_1$. In this case (12) will be understood in the following to be shorthand for an operator-eigenvalue equation. (10) is sufficient as well as necessary for path independence because all paths originate from the 'same point' at $-\infty$ by definition. We note that

$$\oint_C \mathrm{d}\xi_\mu A_\mu(\xi) = 0 \tag{13}$$

is sufficient but not necessary for the formalism to be path independent. (13) holds only if $A_{\mu|\nu} - A_{\nu|\mu} = 0$ which is the uninteresting case with electromagnetic field zero. For nature to satisfy (12) for arbitrary space-like contour C requires quantized behaviour of some sort. A physical example which shows that behaviour like this is possible without the introduction of magnetic monopoles is provided by the quantization of flux in superconductors. In this case, the flux quanta are just given by (12) with $A_0 = 0$ and $q_1 = 2e$, the charge on a Cooper pair. If the phase were a continuous functional of C, n would have to be zero on the right of (12) because the integral would vanish as C shrinks to a vanishingly small loop.

Now consider a second charge q_2 associated with the field $\Psi(x', \mathrm{P}')$ with the field point given by x'. The field variables are defined analogously to those for q_1, ie

$$\Psi(x', \mathrm{P}') = \psi(x') \exp\left(-q_2 \int_{\substack{\mathrm{P}' \\ -\infty}}^{x'} \mathrm{d}\xi_\mu A_\mu(\xi)\right). \tag{14}$$

If we insist that this is also path independent we must have

$$\Psi(x', \mathrm{P}'_1) = \Psi(x', \mathrm{P}'_2) \tag{15}$$

for all space-like P'_1 and P'_2. In particular let us choose P'_1 the same as P_1 from $-\infty$ to point x_3 from which P'_1 then proceeds to the field point x'. P'_2 will similarly coincide with P_2 as far as x_3 and then proceed to x' (see figure 2). Repeating the analysis above

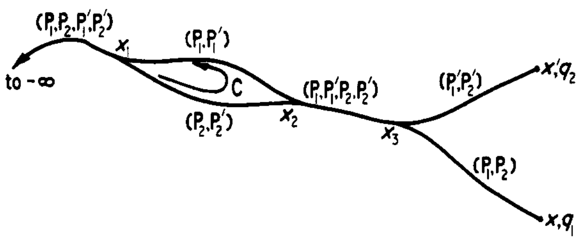

Figure 2. Diagram showing the paths P_1, P_2, P'_1, and P'_2. Charge q_1 is associated with the field $\Phi(x, \mathrm{P})$ and charge q_2 is associated with the field $\Psi(x', \mathrm{P}')$. Various segments of the curve are labelled with symbols in parentheses denoting those paths which follow that segment. x_1, x_2, x_3, x, and x' denote points.

then gives the result that (15) holds if and only if

$$q_2 \oint_C d\xi_\mu A_\mu(\xi) = \pm 2\pi m \tag{16}$$

where m is an integer. The integral in (16) over the closed path C is the same as the integral in (12) from the way we chose our paths. (12) and (16) however imply that for two arbitrary charges q_1 and q_2 we must have

$$\frac{q_1}{q_2} = \pm \frac{n}{m} \tag{17}$$

where n and m are integers. (17) implies that any arbitrary charge can be written as an integer times some fundamental unit of charge e. Thus charge must be quantized with

$$q_1 = \pm ne$$

and

$$q_2 = \pm me. \tag{18}$$

Furthermore (17) requires that charges of both sign exist. Thus charge is quantized if quantum electrodynamics is assumed to be manifestly gauge independent and path independent. We have not shown the converse.

3. Discussion

It is interesting to note that it has been proven that electric charge must be quantized if a magnetic monopole exists anywhere in the universe. This idea originated with Dirac (1931) and has received considerable attention since (see Ross 1969 for further references). Let us look at this magnetic monopole work briefly since it is closely associated with our work above.

Dirac (1931, 1948) formulated the theory of magnetic monopoles in terms of a string of singularities in the vector potential extending from the magnetic monopole to infinity. Requiring the wavefunction of a particle of charge e to be well defined when the gauge associated with the vector potential of the magnetic pole g is transformed leads to a quantization of the product eg. Necessary for this is the requirement that a string never pass through a charged particle, the so-called Dirac veto. Cabibbo and Ferrari (1962) have formulated magnetic monopoles in terms of Mandelstam's path dependent field quantities with no strings of singularities present. They introduce path dependent field quantities for both the electric and magnetic charges. For self-consistency (11) above must be independent of the surface S. This gives

$$\exp\left(-\frac{ie}{2}\int_{\text{closed S}} F_{\mu\nu}\, d\sigma^{\mu\nu}\right) = 1 \tag{19}$$

which leads again to quantization of eg using Gauss's theorem if $\partial^\nu \tilde{F}_{\mu\nu} = g_\mu$ where $\tilde{F}_{\mu\nu}$ is the dual tensor and g_μ the current associated with the magnetic monopoles. The Dirac veto becomes the statement that the paths associated with electric particles cannot cross those associated with the magnetic particles. Cabibbo and Ferrari's work can be made consistent if this requirement is made (Wentzel 1966, Ross 1969). In our work above the assumption of path independence gives

$$\exp\left(-\frac{\mathrm{i}q_1}{2}\int_{\text{open S}} F_{\mu\nu}\,\mathrm{d}\sigma^{\mu\nu}\right) = 1 \tag{20}$$

from (11). (20) leads directly to (10) and to charge quantization as above. (20) is a more severe mathematical requirement than (19) but does not require the existence of magnetic monopoles to give charge quantization. The present work thus shows that the requirement of the existence of a magnetic monopole can be replaced with the assumption of path independence of the formalism. This is an attractive possibility since magnetic monopoles have never been discovered. We require of nature that she satisfies (12) with the proper quantum behaviour rather than that she produces a magnetic monopole.

The important relation (12) which holds if and only if the theory is path independent is not easily amenable to experimental verification. The reason is that (12) involves an integral around a closed *space-like* path. Physical particles such as electrons, on the other hand, always follow *time-like* paths. If (12) is ever found to be violated for any closed space-like path C and any arbitrary electromagnetic field, path independence will be disproven. This is a stringent requirement.

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